JOURNAL OF COMPUTATIONAL PHYSICS 78, 305–312 (1988)

Efficient Transformation of Certain Singular Polynomial Matrix Eigenvalue Problems

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Received April 8, 1987; revised December 7, 1987

Several transformations of the singular $(|\mathbf{A}_r| = 0)$ $n \times n$ polynomial matrix eigenvalue problem $(\mathbf{A}_r \lambda^r + \mathbf{A}_{r-1} \lambda^{r-1} + \cdots + \mathbf{A}_1 \lambda + \mathbf{A}_0)\mathbf{x} = \mathbf{0}$ of degree r, where $\mathbf{A}_r$ has $k < n$ nonzero rows, are described. The transformed problems are either of degree one, order $r(n-1)+k$, and usually (in a sense made precise) nonsingular, or of degree $r-1$, order $n+k$, and singular. For a wide range of k, n, and r, the transformed problems can be solved more efficiently than the original problem of degree r and order n. © 1988 Academic Press, Inc.

I. INTRODUCTION

The problem of determining the eigenvalues λ and eigenvectors $\mathbf{x}$ of the polynomial matrix eigenvalue problem

$$(\mathbf{A}_r \lambda^r + \mathbf{A}_{r-1} \lambda^{r-1} + \cdots + \mathbf{A}_1 \lambda + \mathbf{A}_0)\mathbf{x} = \mathbf{0} \tag{1}$$

arises in many contexts and has been previously discussed by many authors [1–9]. The square matrices $\mathbf{A}_0, ..., \mathbf{A}_r$ have order n, and r is said to be the degree of (1), which we take to be at least two.

In the case where all of the eigenvalues of (1) are needed, the standard procedure is to transform (1) into an order rn linear matrix eigenvalue problem

$$(\mathbf{B}\lambda + \mathbf{C})\mathbf{x} = \mathbf{0} \tag{2}$$

as discussed by several authors [1, 5–8]. Recently, Bridges and Morris [9] have

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0021-9991/88 $3.00

shown how (1) with $|\mathbf{A}_r|=0$ and $|\mathbf{A}_0|\neq 0$ can be recast as a nonsingular problem of order n in the form of (1).

As discussed in [9], (1) frequently arises from the discretization of an ordinary differential equation eigenvalue problem. Bridges and Morris consider the case in which, owing to the appearance of a higher power of the eigenvalue in the differential equation than in some or all of the boundary conditions, $\mathbf{A}_r$ has only a few rows of zeros. They present several transformations of (1) which are appropriate to the case where the number $(n-k)$ of zero rows in $\mathbf{A}_r$ is small. One method transforms (1) to a singular problem of degree one and order rn [cf. (2)], and another transformation converts (1) to a nonsingular (if $|\mathbf{A}_0|\neq 0$) problem of degree r and order n. The former transformation was discussed earlier by Peters and Wilkinson [8].

There are, however, a number of physical problems in which (1) results from the discretization of an ordinary differential equation eigenvalue problem and where, owing to the appearance of a higher power of the eigenvalue in the boundary conditions than in the differential equation itself, the matrix $\mathbf{A}_r$ has only a few nonzero rows [9–13]. A typical example is the Orr–Sommerfeld equation

$$[(d^2/dz^2-\alpha^2)^2+i\alpha R\{d^2U/dz^2-(U-c)(d^2/dz^2-\alpha^2)\}]\phi=0 \tag{3a}$$

with boundary conditions

$$\phi(-1)=d\phi(-1)/dz=0 \tag{3b)-(3c}$$

$$(c-1)\,d^2\phi(0)/dz^2+[\alpha^2(c-1)-2]\,\phi(0)=0 \tag{3d}$$

$$\begin{aligned}(2\cot\beta+\alpha^2SR)\phi(0)+[R(c-1)^2+3i\alpha(c-1)]\,d\phi(0)/dz\\ -i(c-1)\,d^3\phi(0)/dz^3=0\end{aligned} \tag{3e}$$

governing the linear stability of a viscous fluid flowing with velocity profile $U(z)$ down a plane inclined at an angle β. Here, the wave speed c is the eigenvalue to be computed for fixed values of the wavenumber α, Reynolds number R, and surface tension parameter S [11]. Discretization using either finite difference or spectral techniques gives a matrix eigenvalue problem

$$(\mathbf{D}_2\lambda^2+\mathbf{D}_1\lambda+\mathbf{D}_0)\,\mathbf{x}=\mathbf{0}, \tag{4}$$

where $\mathbf{D}_2$ has only one nonzero row, regardless of the number of grid points or trial functions used in the discretization.

The standard transformations of (1) with $|\mathbf{A}_r|=0$ produce either a singular eigenvalue problem of degree one and order rn [8], or if $|\mathbf{A}_0|\neq 0$, either a nonsingular problem of degree r and order n or a nonsingular problem of degree one and order rn [8, 9].

In what follows, we present transformations of (1) into versions of (1) and (2) that, by use of existing techniques, can be solved more efficiently than either the original version of (1) or the versions of (2) produced by previous transformation methods. We discuss the quadratic case (4) in some detail in Section II and present the results for the more general case of (1) in Section III.

II. Quadratic Case ($r=2$)

More complicated problems than (3a)–(3e) (see [12]) generally give rise to matrices $\mathbf{A}_2$ in (1) with more than one nonzero row, so in the sequel we will consider the case where k $(1 \leqslant k < n)$ of the rows of $\mathbf{A}_2$ are nonzero.

We begin by observing that when $\mathbf{A}_2$ has k nonzero rows, the maximum possible degree of the determinantal polynomial of

$$|\mathbf{A}_2\lambda^2 + \mathbf{A}_1\lambda + \mathbf{A}_0|$$

is $n+k$. It thus seems reasonable to seek a transformation of (4) into a nonsingular $(n+k)$th order version of (2).

We partition $\mathbf{A}_2$ as

$$\mathbf{A}_2 = \left[\begin{array}{c} \mathbf{0}_1 \\ \hline \mathbf{b} \end{array}\right],$$

where $\mathbf{b}$ and $\mathbf{0}_1$ are $k \times n$ nonzero and $(n-k)\times n$ zero matrices, respectively. We define a $k \times 1$ vector

$$\mathbf{y} = \lambda \mathbf{b}\mathbf{x} \tag{5}$$

and rewrite (1) as

$$\lambda \left[\begin{array}{c} \mathbf{0}_2 \\ \hline \mathbf{y} \end{array}\right] + \mathbf{A}_1\lambda\mathbf{x} + \mathbf{A}_0\mathbf{x} = \mathbf{0}, \tag{6}$$

where $\mathbf{0}_2$ is an $(n-k)$-vector of zeros. We observe that (5) and (6) are linear in λ, so we can combine them to get

$$\left[\begin{array}{c|c} \mathbf{A}_1\lambda + \mathbf{A}_0 & \begin{array}{c} \mathbf{0}_3 \\ \lambda\mathbf{I}_k \end{array} \\ \hline \lambda\mathbf{b} & -\mathbf{I}_k \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{y} \end{array}\right] = \mathbf{0}, \tag{7}$$

where $\mathbf{0}_3$ is an $(n-k)\times k$ matrix of zeros, and $\mathbf{I}_k$ is the kth order identity matrix. Now (7) can be rewritten as

$$(\mathbf{F}_1\lambda+\mathbf{F}_0)\mathbf{V}=\mathbf{0}, \tag{8}$$

where

$$\mathbf{V}=[\mathbf{x}\quad \mathbf{y}]^T,$$

$$\mathbf{F}_1=\left[\begin{array}{c|c} \mathbf{A}_1 & \begin{array}{c}\mathbf{0}_3\\ \hline \mathbf{I}_k\end{array} \\ \hline \mathbf{b} & \mathbf{0}_4 \end{array}\right],$$

$$\mathbf{F}_0=\left[\begin{array}{c|c} \mathbf{A}_0 & \mathbf{0}_5 \\ \hline \mathbf{0}_5^T & -\mathbf{I}_k \end{array}\right],$$

and $\mathbf{0}_4$ and $\mathbf{0}_5$ are, respectively, $k\times k$ and $n\times k$ matrices of zeros.

Thus, (8) is of the form (2), has order $n+k<2n$, and will be singular only if the $n\times n$ matrix

$$\mathbf{P}=\left[\begin{array}{c} \mathbf{M} \\ \hline \mathbf{b} \end{array}\right]$$

is singular, where $\mathbf{M}$ is an $(n-k)\times n$ matrix consisting of the first $n-k$ rows of $\mathbf{A}_1$. Thus, $\mathbf{P}$ [and hence (8)] will be singular only for some set of pairs of matrices $\mathbf{A}_1$ and $\mathbf{b}$ of measure zero. In contrast, the standard transformation of (4) into (2) for $|\mathbf{A}_2|=0$ has final order $2n$ and is always singular.

III. General Case ($r \geqslant 3$)

In the general case ($r\geqslant 3$), we consider two approaches, each of which yields a result that reduces to (8) when $r=2$. The first approach leads to an eigenvalue problem of degree one and order $(r-1)\,n+k$, while the second yields a problem of degree $r-1$ and order $n+k$.

IIIa. *Transformation to an Eigenvalue Problem of Degree One and Order* $(r-1)n+k$

We begin by defining $\mathbf{x}=\mathbf{y}_0$ and writing

$$\lambda^j \mathbf{y}_0=\mathbf{y}_j, \qquad 1 \leqslant j \leqslant r-2, \tag{9}$$

with each $\mathbf{y}_j$ an n-vector. We can then write (1) as

$$\left[\begin{array}{c} \mathbf{0}_3 \\ \hline \lambda \mathbf{y}_{r-1} \end{array}\right] + \mathbf{A}_{r-1}\lambda\mathbf{y}_{r-2}+\mathbf{A}_{r-2}\mathbf{y}_{r-2}+\cdots+\mathbf{A}_1\mathbf{y}_1+\mathbf{A}_0\mathbf{y}_0=\mathbf{0}, \tag{10}$$

where

$$\mathbf{y}_{r-1}=\lambda^{r-1}\mathbf{b}\mathbf{y}_0=\lambda\mathbf{b}\mathbf{y}_{r-2}. \tag{11}$$

We then combine (9)–(11) to obtain

$$(\mathbf{G}_1\lambda+\mathbf{G}_0)\mathbf{W}=\mathbf{0}, \tag{12}$$

where $\mathbf{W}=[\mathbf{y}_0 \quad \mathbf{y}_1 \quad \cdots \mathbf{y}_{r-2} \quad \mathbf{y}_{r-1}]^T$ is an $[(r-1)n+k]\times 1$ column vector and

$$\mathbf{G}_1=\left[\begin{array}{c|cc} \mathbf{I}_{n(r-2)} & \multicolumn{2}{c}{\mathbf{0}_6} \\ \hline \multirow{2}{*}{$\mathbf{0}_6^T$} & \mathbf{A}_{r-1} & \begin{array}{c}\mathbf{0}_3\\ \hline \mathbf{I}_k\end{array} \\ \cline{2-3} & \mathbf{b} & \mathbf{0}_4 \end{array}\right]$$

and

$$\mathbf{G}_0=\left[\begin{array}{c|c|c|c|c} \multicolumn{4}{c|}{\mathbf{H}} & \mathbf{0}_7 \\ \hline \mathbf{A}_0 & \mathbf{A}_1 & \cdots & \mathbf{A}_{r-2} & \mathbf{0}_5 \\ \hline \multicolumn{3}{c|}{\mathbf{0}_7^T} & \mathbf{0}_5^T & -\mathbf{I}_k \end{array}\right],$$

where $\mathbf{I}_{n(r-2)}$ is the $(r-2)n \times (r-2)n$ identity matrix, $\mathbf{H}$ is an $(r-2)n \times (r-1)n$ matrix with elements

$$h_{ij} = \begin{cases} -1, & j-i=n, \\ 0, & j-i \neq n, \end{cases}$$

and $\mathbf{0}_6$ and $\mathbf{0}_7$ are, respectively, $(r-2)n \times (n+k)$ and $(r-2)n \times k$ matrices of zeros.

As before, $\mathbf{G}_1$ is singular only for a set of pairs of $\mathbf{A}_{r-1}$ and $\mathbf{b}$ of measure zero, i.e., only when the $n \times n$ matrix

$$\mathbf{E} = \begin{bmatrix} \mathbf{J} \\ \hline \mathbf{b} \end{bmatrix}$$

is singular, where $\mathbf{J}$ is an $(n-k) \times n$ matrix consisting of the first $n-k$ rows of $\mathbf{A}_{r-1}$. The order of (12) is $(r-1)n+k$, which again corresponds to the maximum degree of the characteristic polynomial of (1).

IIIb. *Transformation to an Eigenvalue Problem of Degree $r-1$ and Order $(n+k)$*

A transformation of (1) to another form of (1) with degree $r-1$ and order $(n+k)$ is suggested by the transformations presented in Sections II and IIIa. We define $\mathbf{y}$ as in (5) and rewrite (1) as

$$\lambda^{r-1} \begin{bmatrix} \mathbf{0}_2 \\ \hline \mathbf{y} \end{bmatrix} + (\mathbf{A}_{r-1}\lambda^{r-1} + \cdots + \mathbf{A}_1\lambda + \mathbf{A}_0)\mathbf{x} = \mathbf{0}. \tag{13}$$

We define $\mathbf{V}$ as in Section II, and combine (5) and (13) to obtain

$$(\mathbf{L}_{r-1}\lambda^{r-1} + \mathbf{L}_{r-2}\lambda^{r-2} + \cdots + \mathbf{L}_1\lambda + \mathbf{L}_0)\mathbf{V} = \mathbf{0}, \tag{14}$$

where

$$\mathbf{L}_{r-1} = \left[\begin{array}{c|c} \mathbf{A}_{r-1} & \begin{array}{c} \mathbf{0}_3 \\ \hline \mathbf{I}_k \end{array} \\ \hline \mathbf{0}_5^T & \mathbf{0}_4 \end{array}\right],$$

$$\mathbf{L}_j = \left[\begin{array}{c|c} \mathbf{A}_j & \mathbf{0}_5 \\ \hline \mathbf{0}_5^T & \mathbf{0}_4 \end{array}\right] \qquad (2 \leqslant j \leqslant r-2),$$

$$\mathbf{L}_1 = \left[\begin{array}{c|c} \mathbf{A}_1 & \mathbf{0}_5 \\ \hline \mathbf{b} & \mathbf{0}_4 \end{array}\right],$$

and

$$\mathbf{L}_0 = \left[\begin{array}{c|c} \mathbf{A}_0 & \mathbf{0}_5 \\ \hline \mathbf{0}_5^T & -\mathbf{I}_k \end{array}\right].$$

Thus, (14) is an eigenvalue problem of degree $r-1$ and order $n+k$, which can be solved using the techniques developed by Bridges and Morris [9]. The advantage of (14) over the originial form (1) is the reduction of degree at the expense of increasing the order from n to $(n+k)$.

IV. Comparison to Other Methods

The transformations described herein employ only store and fetch operations, so that their efficiencies can be judged by directly comparing the cost of solving the original and transformed eigenvalue problems.

For the comparison, we consider two methods for solving the original and transformed eigenvalue problems [9]: (1) transformation to a matrix eigenvalue problem of degree one (e.g., the companion matrix), with solution by the QZ algorithm, and (2) direct solution of a polynomial matrix eigenvalue problem by the Bezout factorization, as described by Bridges and Morris.

For method (1), the operations count is $O(M^3)$, where M is the order of the first degree matrix eigenvalue problem. For the companion matrix transformation, we have $M=rn$. Thus, transformation to a problem of degree one and order $M=(r-1)n+k \leqslant rn$ is always preferred over the companion matrix transformation. Transformation to an $(r-1)$th degree problem of order $n+k$, followed by the companion matrix transformation, is competitive only for $r=2$, when $M=(r-1)(n+k)=(r-1)\,n+k$.

For method (2), the operations count appears to be $O(R^2N^3)$, where $R \geqslant 2$ and N are the degree and order, respectively, of the polynomial matrix eigenvalue problem to be solved *via* the Bezout factorization. The ratio of the operations counts for the $(r-1)$th degree problem of order $n+k$ and the original problem is then

$$\frac{C_{\mathrm{IIIb}}}{C_{\mathrm{orig}}} = \left(1 - \frac{1}{r}\right)^2 \left(1 + \frac{k}{n}\right)^3.$$

Thus, the transformation of Section IIIb should be performed prior to application of the direct solution technique of Bridges and Morris when

$$\frac{k}{n} < \left(\frac{r}{r-1}\right)^{2/3} - 1.$$

Acknowledgments

This work was supported in part at UCLA by NSF Grant MEA 82-04944 to Professor R. E. Kelly. The work of the first author was also supported in part by NSF Grant MSM-8451157 and by the Union Oil Company of California.

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